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DISCUSSION OF REINFORCED CONCRETE SKEWED RIGID-FRAME AND ARCH BRIDGES

(Published in April, 1950)

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STRUCTURAL DIVISION

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DISCUSSION

D. A. NETTLETON,[10] Assoc. M. ASCE.—An examination of Fig. 1, Eqs. 6, and Table 2 will show that the expressions for the indeterminate reactions are based on deformations from bending moments acting in planes perpendicular to the line of the abutments, and torsional moments acting in planes parallel to the line of the abutments. Also, on any plane cut through the structure parallel to the abutments, it is assumed that the moments (and therefore the deformations that determine the reactions) are the same on each unit of the width of the structure. The same assumptions as to bending and torsional moments were made by J. Charles Rathbun, M. ASCE, in setting up his equations,[2] published in 1924, and they have formed the basis for several papers since that date.

The writer questions the validity of these assumptions in regard to bending and torsional moments. In the vicinity of the sides of the deck, the bending moments must act in planes parallel to the sides and the torsional moments in planes perpendicular to the sides. For a rigid-frame bridge, narrow in relation to its span, this type of moment action applies fairly closely across the width of the deck. The writer has set up equations for the indeterminate reactions on the basis of this moment action. To compare the results obtained by the author's method (hereafter called Method A) and the writer's method (hereafter called Method B), the two have been applied to rigid-frame bridges with skew angles of 0°, 30°, and 60°. To simplify the calculations, the frame is assumed to be of constant thickness throughout the legs and deck, and the deck is assumed to be in a horizontal plane. The span, in a direction perpendicular to the line of the abutments is called L, and the height of the legs is taken as $0.25\ L$.

A comparison between the two methods is afforded by Table 10. The deck moments in Table 10(a) act in a plane parallel to the sides of the deck. For Method A, these values are obtained from moments acting in planes perpendicular to the line of the abutments by multiplying by the square of the secant of the skew angle. This corresponds to the author's "turning process" which he applies to the area of reinforcing required. "Turning" the moments instead of the required areas of reinforcing results in the same requirements for reinforcing, and is done here to facilitate a comparison between the two methods.

Method B gives larger knee moments for deck loading, and that, of course, results in smaller moments at the center of the span. In the case of temperature change, Method B gives very materially smaller knee moments, particularly in the deck. Deck moments by Method B are only 58% of the deck moments by Method A for a 30° skew, and 7% for a 60° skew. Neither method shows the variation in moment across the width of the section, and neither method yields any information as to the transverse moments.

NOTE.—This paper by Maurice Barron was published in June, 1950. The numbering of footnotes, illustrations, equations, and tables in this separate are a continuation of the consecutive numbering used in the original paper.

[10] Supervising Urban Engr., State Highway Dept., Dallas, Tex.

[2] "Analysis of the Stresses in the Ring of a Concrete Skew Arch," by J. Charles Rathbun, *Transactions*, ASCE, Vol. LXXXVII, 1924, p. 611.

V. P. Jensen,[11] M. ASCE, has made an exact analysis for a skewed slab of constant section by the method of difference equations. A method such as this would show the stresses in a rigid-frame bridge but, to the writer's knowledge, no one has been successful in applying it to a rigid-frame bridge.

TABLE 10.—Relative Moments in a Flat-Topped Frame of Constant Thickness

Quantity	Skew angle θ	Uniform Load of w Pounds per Square Foot		Temperature Rise of $t°$	
		Method A	Method B	Method A	Method B
		(*a*) Knee Moments			
Deck	0°	$-0.0714\ w\ L^2$	$-0.0714\ w\ L^2$	$-\ 3.42\ E\ I\ c\ t/L$	$-\ 3.42\ E\ I\ c\ t/L$
	30°	$-0.095\ w\ L^2$	$-0.102\ w\ L^2$	$-\ 6.05\ E\ I\ c\ t/L$	$-\ 3.5\ E\ I\ c\ t/L$
	60°	$-0.286\ w\ L^2$	$-0.330\ w\ L^2$	$-\ 54.8\ E\ I\ c\ t/L$	$-\ 3.81\ E\ I\ c\ t/L$
Leg	0°	$-0.0714\ w\ L^2$	$-0.0714\ w\ L^2$	$-\ 3.42\ E\ I\ c\ t/L$	$-\ 3.42\ E\ I\ c\ t/L$
	30°	$-0.0714\ w\ L^2$	$-0.0732\ w\ L^2$	$-\ 4.55\ E\ I\ c\ t/L$	$-\ 3.93\ E\ I\ c\ t/L$
	60°	$-0.0714\ w\ L^2$	$-0.0789\ w\ L^2$	$-\ 13.7\ E\ I\ c\ t/L$	$-\ 5.11\ E\ I\ c\ t/L$
		(*b*) Reactions at the Bottom of the Legs			
R_z	0°	$+0.286\ w\ L$	$+0.286\ w\ L$	$+\ 13.68\ E\ I\ c\ t/L^2$	$+13.68\ E\ I\ c\ t/L^2$
	30°	$+0.286\ w\ L$	$+0.293\ w\ L$	$+\ 18.2\ E\ I\ c\ t/L^2$	$+15.72\ E\ I\ c\ t/L^2$
	60°	$+0.286\ w\ L$	$+0.316\ w\ L$	$+\ 54.7\ E\ I\ c\ t/L^2$	$+20.4\ E\ I\ c\ t/L^2$
R_x	0°	0	0	0	0
	30°	$+0.165\ w\ L$	$+0.169\ w\ L$	$+\ 17.25\ E\ I\ c\ t/L^2$	$+15.81\ E\ I\ c\ t/L^2$
	60°	$+0.496\ w\ L$	$+0.548\ w\ L$	$+115\ E\ I\ c\ t/L^2$	$+55.7\ E\ I\ c\ t/L^2$
M_z	0°	0	0	0	0
	30°	$+0.0069\ w\ L^2$	$+0.0077\ w\ L^2$	$-\ 4.31\ E\ I\ c\ t/L$	$-\ 4.68\ E\ I\ c\ t/L$
	60°	$+0.0206\ w\ L^2$	$+0.0081\ w\ L^2$	$-\ 28.7\ E\ I\ c\ t/L$	$-14.67\ E\ I\ c\ t/L$
M_y	0°	0	0	0	0
	30°	0	0	$+\ 3.37\ E\ I\ c\ t/L$	$+\ 3.37\ E\ I\ c\ t/L$
	60°	0	0	$+\ 10.13\ E\ I\ c\ t/L$	$+10.13\ E\ I\ c\ t/L$

The design of the transverse reinforcement should be controlled by the transverse moments, and not by the torsional moments, because the torsional moments are generally so small that no transverse reinforcing is required. The analysis of skewed slabs by Mr Jensen may be used as a rough guide to the amount of transverse reinforcement required in the deck. This analysis shows the following maximum transverse moments acting in a plane at right angles to the line of the skew span:

Skew	Transverse M
0°	$0.0085\ w\ l^2$
30°	$0.0148\ w\ l^2$
45°	$0.0193\ w\ l^2$
60°	$0.0241\ w\ l^2$

in which w is the uniform load per square foot, and l is the skew span.

The transverse moments in the central part of the span of a rigid-frame bridge may be obtained, roughly, by using these moment coefficients, taking l as the skew distance between points of zero moment for dead load and w as

[11] "Analyses of Skew Slabs," by V. P. Jensen, *Engineering Experiment Station Bulletin Series No. 332*, Univ. of Illinois, Urbana, Ill., 1941, Tables 5 to 8, inclusive.

sum of the dead load and the average live load. In addition to this moment, there is a transverse moment,[12] due to concentrated wheel loads, equal to 0.3 P approximately, in which P is the concentrated load, in pounds, and the moment is in units of foot-pounds per foot. Since the transverse rods generally run parallel to the abutments instead of perpendicular to the skew span, this steel is not fully effective for transverse moments and, therefore, the required areas must be multiplied by the square of the secant of the skew angle. These transverse moments produce tension in the bottom of the slab, and therefore should be used to determine the transverse bottom steel. The transverse top steel may be made somewhat less.

ERNEST H. HARDER,[13] M. ASCE.—Aside from the application of the author's method to double-span bridges, this paper is very much like that of Richard M. Hodges[4] except, perhaps, in the method of demonstration. Any attempt to simplify the lengthy exposition of the original paper by J. Charles Rathbun,[2]

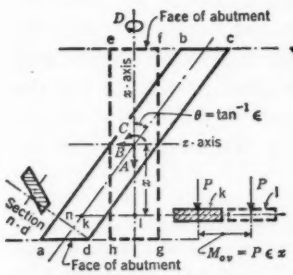

FIG. 15

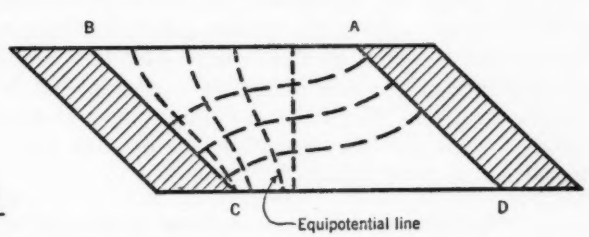

FIG. 16

M. ASCE, is commendable. In the discussion of that paper, the writer demonstrated[14] that the solution of the reactions of any skewed arch bridge consisted of two parts—the solution of the reactions of a right arch and the solution of the effect of the skew on that right arch, provided that the x-axis is chosen perpendicular to the line of the abutments and the y-axis and z-axis are at right angles to the x-axis all through the crown of the arch.

It seems to the writer that it is time for a review of the problem that has been obscured by endless pages filled with integral signs, double subscript deltas, changes of notation, etc. Having selected, as a matter of convenience, the x-axis perpendicular to the face of the abutments and y-axis and z-axis at right angles to the x-axis at the center of the crown, the designer can erect about these axes a dummy arch (e-f-g-h, Fig. 15) which has the same curvature and thickness as the skew arch a-b-c-d when viewed in the direction of the z-axis. Then, whether the skew arch and the dummy arch are cut through the crown

[12] *Public Roads*, March, 1930, p. 8, Fig. 6.

[13] Cons. Structural Engr., Newark, N. J.

[4] "Simplified Analysis of Skewed Reinforced Concrete Frames and Arches," by Richard M. Hodges, *Transaction*, ASCE, Vol. 109, 1944, p. 913.

[14] Discussion by Ernest H. Harder of "Analysis of the Stresses in the Ring of a Concrete Skew Arch," by J. Charles Rathbun, *ibid.*, p. 660.

along the z-axis, or in any other manner, to create simple systems, any load such as P at point k may be transferred to the dummy arch at point l by an equal load P and a twisting moment $P \epsilon x$. Thus, the designer can operate on the dummy arch as on any right arch to obtain the thrusts, shears, and moments due to the vertical loads P and he can find the shears parallel to the z-axis, the torsions on any radial section, and the moments on the radial section which the author calls M_u. These shears and moments are in reality treated like secondary forces on the dummy arch. This is what the writer proposed in 1924 and it is the same as the author is proposing in 1950. In effect both methods are the same—merely the method of presentation is different. For purposes of illustration, the writer placed load P on the center line of the skew arch. A load can be placed anywhere on the skew arch and its secondary effects on the dummy arch are found in the same manner. It is also immaterial whether the ends of the arches are hinged or fixed.

By the usual methods of summation and using the areas and moments of inertia of the various sections of the dummy arch (all of which are still somewhat questionable), the several indeterminate reactions that hold the primary and secondary forces of the dummy arch in equilibrium can be found. Having found these reactions for the dummy arch, the designer is finished with it and any attempt to find moments, shears, thrusts, and torsions at section l of the dummy arch and then attempt to apply them at section k of the skew arch is wrong. That is what the author has done in Eq. 17, for instance. As the writer has stated elsewhere:[15]

> "Before these forces and torsions can be used to determine the stresses in the arch, T_x and T_y must be combined and the components of the resultant along the center line of the bridge at right angles to this axis must then be determined. This should also be done in connection with the bending moments and torsions so that the arch stresses can be found for sections at right angles to the faces of the arch."

So far, the only experimental substantiations of the theoretical methods are the tests made by the late George E. Beggs, M. ASCE, for the Special Committee on Concrete and Reinforced Concrete arches in 1926. In reviewing this matter in the final report of the committee neither the text nor the figures are too explicit regarding the direction of the coordinate axes. The figures, however, indicate that for both the experimental arch, and the theoretical one, the x-axis was parallel to the face of the skew arch whereas, in the past, all writers have chosen the x-axis perpendicular to the face of the abutment for theoretical computations. Since this writer is suggesting a review of the entire problem, perhaps this matter may be clarified.

In addition to these mistaken ideas of the general problem, there has been a tendency toward simplification. In some instances, it appears that the simplification has been over done, for example, neglecting M_u, which is a moment that causes the normal thrust T_v to depart more and more toward the obtuse corner of the arch. That there is a concentration of stress at the obtuse corner of the arch is well known. Such a concentration of stress was the cause of the failure

[15] Discussion by Ernest H. Harder of "Analysis of the Stresses in the Ring of a Concrete Skew Arch," by J. Charles Rathbun, *Transactions*, ASCE, Vol. LXXXVII, 1924, p. 672.

of a skew arch bridge in Bendigo, Australia. Clyde T. Morris,[16] M. ASCE, clearly showed such stress concentration in the test of a model reinforced concrete arch having a span of 4 ft. Perhaps there is no great stress concentration in the obtuse corner of a rigid-frame span because of the flatness of the arch but then, Mr. Barron's paper, by its title, includes arch bridges. A word of caution is advisable.

Writers seem to lay great stress on the computation of the indeterminate reactions, but not enough effort and thought have been given to the equally important problem of what to do with these reactions after they have been computed. Only one writer, Bernard L. Weiner,[17] M. ASCE, has gone into that subject extensively. Mr. Weiner has evolved a long, complicated, and highly technical exposition of the subject which renders it difficult to use for design purposes. A brief exposition of this subject is found in a later paper by Mr. Rathbun.[18] Mr. Rathbun follows the lines that this writer prefers—namely, transposing the reactions, as computed, into components parallel to the skew center line and at right angles thereto. For example, Mr. Rathbun has chosen to resolve T_x and T_y into a component T acting along the center line of the skew span and s, a shear, acting at right angles to the fascia. He stops there, however, and gives a method for distributing only the shear s to transverse reinforcing steel, which is similar to that method given in the paper by Mr. Barron. In the design of sections, Mr. Barron uses an oversimplified method which is also mentioned in the paper by Mr. Hodges. This method is called the "turning process" for reinforcement and involves a serious error. He virtually assumes that moment M_L and thrust T_L are acting on the diagonal section parallel to the abutments. After all, this diagonal plane is a section through the right dummy arch, and all that remains is to increase the computed steel areas by multiplying by sec V on the theory that the right section across the skew is shorter than the diagonal section. What about the last diagonal section at the face of the abutment? How can one justify the "turning process" there?

If one chooses to turn the entire coordinate axis system so that the x-axis falls along the center line of the skew arch, then the new components must act on the rather complex sections mentioned by the writer in his 1924 discussion,[19] and the last of such sections lies across the skew arch axis starting at the obtuse angle (Fig. 15). This step leaves a triangular area a-d-n in which the stresses cannot be found so readily. If the arch is very wide compared to its span length, there may be only one rectangular section at the crown—the rest is a triangular area. The writer has always felt that, for extremes of skew and width, the method proposed by Mr. Rathbun does not hold.

In the design of transverse reinforcement, as given by Mr. Barron and which is similar to that given by Mr. Rathbun in his 1924 paper (and also in that of Mr. Hodges), Mr. Barron, as well as Mr. Hodges, seems to make an error by finding the areas of steel needed for a length of arch of 1 in. On the other hand,

[16] "Thrust of Skew Barrel Arch Measured on Laboratory Model," by Clyde T. Morris, *Engineering News-Record*, April 20, 1922, p. 638.

[17] "The Design of a Reinforced Concrete Skew Arch," by Bernard L. Weiner, *Transactions*, ASCE, Vol. 96, 1932, p. 1212.

[18] "An Analysis of Multiple-Skew Arches on Elastic Piers," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 98, 1933, p. 12.

[19] Discussion by Ernest H. Harder of "Analysis of the Stresses in the Ring of a Concrete Skew Arch," by J. Charles Rathbun, *ibid.*, Vol. LXXXVII, 1924, p. 664, Fig. 14.

Mr. Rathbun gives his results for a length of 12 in. along the center line of the arch. The demonstrations offered by Mr. Barron in Eqs. 45 to 48 and Fig. 14 are based on a demonstration published in 1929.[9] To conform to that demonstration,[20] the shear values as given in Eq. 47*b* should be substituted for $(v-v')b$ in the usual formula for stirrup spacing.

The writer has no ready solution to offer for his criticism of this paper. What he wrote in 1924 he still affirms today. In reviewing the efforts of others, he believes nothing really new has been injected into the subject.

The only hope for any real advance would seem to lie in the realm of experimentation to ascertain the nature and pattern of stresses in the arch. Whether this could best be done by the electric analogy method or by the brittle lacquer technique is something for others to select. In 1903, the late W. C. Kernot,[21] M. ASCE, professor of engineering, University of Melbourne, Melbourne, Australia (who has been so frequently mentioned in connection with the failure of a skew arch bridge at Bendigo) produced a stress pattern by the electric analogy method. Although it is crude (Fig. 16) it is the first and only stress pattern which the writer has seen.

Mr. Kernot tried another experiment: He endeavored to construct and test an indiarubber model having parallel lines or circles of known diameter drawn upon it, the deformation of which would indicate the direction and magnitude of the stress. This model proved unmanageable so he used a sheet of rubber and suspended loads from it, to obtain elongations instead of compressions. The end results showed great stress at the obtuse corner of the arch. This idea, if carried out in a model of hard material, would suggest the brittle lacquer technique.

EGIDIO O. DI GENOVA,[22] M. ASCE.—The correlation of externally applied loads with internal stresses, as proposed in this paper, is quite ingenious. The fundamental concepts of the elastic theory, here applied, prove once more that, when a structural analyst masters the fundamentals of this wonderful art, he is always at ease in undertaking any problem pertinent to the safest and surest way of designing a structure. He acquires a well founded judgment in deciding what is essential and what may be disregarded when analyzing the importance of certain induced stresses with respect to others.

Mr. Barron shows clearly how many forces can be disregarded, and the writer proposes still further approximations that can be made without impairing the essential safety of a structure. In a work entitled "Elastic Theory Applied to Solids," on file for reference at the Engineering Societies Library,[22a] the writer has treated similar subjects in considerable detail.

A reinforced concrete skewed structure can first be considered as a monolith fixed against excessive partial settlements. In a very general sense, the skewed arch can be compared with a square arch as follows: At any section of a square

[9] "Standards of Design for Concrete," *No. 3Yb*, Bureau of Yards and Docks, U. S. Navy Dept., Washington, D. C., 1929, Section 8-13d.

[20] "Standards of Design for Concrete," *No. 3Yb*, Bureau of Yards and Docks, U. S. Navy Dept., Washington, D. C., 1929, p. 64.

[21] "The Stresses on Skew Arches," by W. C. Kernot, *Engineering News*, June 11, 1903, p. 529.

[22] Water Supply Engr., Board of Water Supply, New York, N. Y.

[22a] 29 West 39th Street, New York 18, N. Y.

arch the total resulting moment is a function of the horizontal component of reaction X, the vertical component Y, the reaction moment M_R (if fixed), and the externally applied forces ΣP (for instance, vertical loads). The relative partial moments are, respectively, $X\,y$, $Y\left(\frac{L}{2}+x\right)$, M_R, and $\Sigma(P\,a_h)$, in which a_h represents the horizontal distance of each load at the left of the section under consideration.

Consider the skewed arch that has a square span L. The coordinates of the center of gravity of each section are x, y, and z, whereas in the square arch they were x and y. Consequently the partial moments at the center of gravity of each section of the skewed arch are: $X'\,y$ and $X'\left(\frac{L}{2}\tan\theta+z\right)$, $Y'\,x$ and $Y'\left(\frac{L}{2}\tan\theta+z\right)$, and $\Sigma\,(P\,a_h)+\Sigma\,(P\,a_h\tan\theta)$—$M_R$ being considered as an infinitesimal force at infinity which does not change.

It is obvious that Y' for the same width of bridge must be larger than Y, and so X' and M'_R are larger than X and M_R. It is already evident that a skewed bridge cannot be designed for an arch of span L, because the moments $X'\,y$, $Y'\,x$, and M_R are larger than the corresponding moments in the square arch.

Concentrating on the other partial moments relative to the skewed span—

$$M = X\left(\frac{L}{2}\tan\theta+z\right) \qquad (49)$$

—may be interpreted as proportional to M_u as defined in the paper in so far as the former acts horizontally whereas the latter is a component of the former, acting in the plane normal to the section under consideration. In the final analysis this moment acts (as the author states) on the full depth of the arch and consequently is not very important with respect to the unit stresses; the other component at an angle of 90° also generates shear on the full depth of the section and, for the same reason, may be considered unimportant. The parameter $Y'\left(\frac{L}{2}\tan\theta+z\right)$ can be partly compared to the M_v of the paper. Considered as its component in the plane of the section, it would generate shearing stresses on the full length of the section, whereas its component at an angle of 90° (normal to the section) would generate fiber stresses due to a moment for which the arm length of the resultant compression and tension is very nearly the depth of the bridge.

Effects similar to the foregoing result from the partial moments $\Sigma(P\,a_h\tan\theta)$ except that the corresponding stresses generally have opposite values because the former are derived from forces of reaction whereas the latter are derived from externally applied loads. In a skewed bridge, the total reaction has a component acting parallel to the z-axis, and the moment of reaction has a component acting in a plane parallel to the (y,z)-plane. Both these components would contribute to the total values of M_u and M_v as given in the paper and as interpreted by the writer.

Thus the writer would minimize the importance of the extra moments which the paper rightfully classifies as secondary stresses; but these moments do exist and may have an effect in increasing the maximum stresses on a skew span. Therefore, it is safe to add to any system of computations a certain margin of safety by increasing the resistance value of this type of arch. This margin of safety is supplied by increasing the span length as the angle θ of the skewed arch increases, for any value of θ.

For simplicity assume a skewed arch bridge with a constant cross section. If L represents the span perpendicular to the faces of the abutments and computations are applied to a strip of arch 1 ft wide and parallel to the faces of the bridge, this arch will have a skew span $L \sec \theta$, which is longer than the right span.

Referring to Fig. 4, the center line may be taken as the center line of a 1-ft rib containing the centers of gravity of the various finite or infinitesimal elastic elements into which this strip may be subdivided. This rib is referred to a system of coordinates similar to the one shown in the paper, the origin of which is on the z-axis, whereas the x-axis and the y-axis are in the vertical plane containing the centers of gravity of all the elastic elements of the rib.

For an arrangement such as this the values of y corresponding to the intersections of the intrados or extrados, with radiating planes (planes perpendicular to the line of the centers of gravity of the rib and consequently to the (x,y)-plane), are not constant—that is, these lines are curved. Consequently the normal sections that result in subdividing the rib into elastic elements of finite lengths Δs are of irregular shape; and they are not symmetrical with respect to the (x,y)-plane. However, the two elastic parts into which they are subdivided by this plane are equal. Furthermore, the trace of each elastic element on the (x,y)-plane represents a segment of arc similar to a segment of a square span of uniform thickness. In other words, the shape of the rib on the (x,y)-plane is the same as that of the developed intersection of the cylindrical vault with a plane parallel to the plane containing one of the faces of the bridge.

This conception of the rib is actually not in strict conformity with the theories of planar elastic arches; but in the realm of approximations, applying those theories to a rib only 1 ft deep, the theory is an admissible and practical possibility.

To be convinced of the reasonable nature of such an approximation one needs only to remember that the surface containing the central strata of fibers or the neutral strata, with respect to the moment stresses, is always parallel to the intrados and extrados (zones of maximum fiber stresses), and consequently the concepts of fiber stresses and of the work of deformation may be considered unaltered.

A larger degree of approximation accrues in the investigation of the plane of intersection of the vault with the abutments. End elements Δs are extremely irregular; that is, the extreme section of each of these end elastic elements is not radial but skewed. In the computation of the magnitude of the total reactions, the approximation is quite close to the actual values because a thorough analysis of any method generally used for computing the reactions of support of a square arch reveals that the formulas are derived from the concept of integrals or of

summations. The influence of one term is not very important in obtaining the value of a force, a moment, or an elastic deformation, especially if mathematical expressions with the same degree of approximation are in the numerator as are in the denominator.

Uncertainties as to stress behavior at the support of this rib can be taken care of in a practical manner by increasing the elastic mass around that section so that it may be considered as a truly fixed support of the vault. In the writer's opinion, the failure to recognize the possible effect of such irregularity

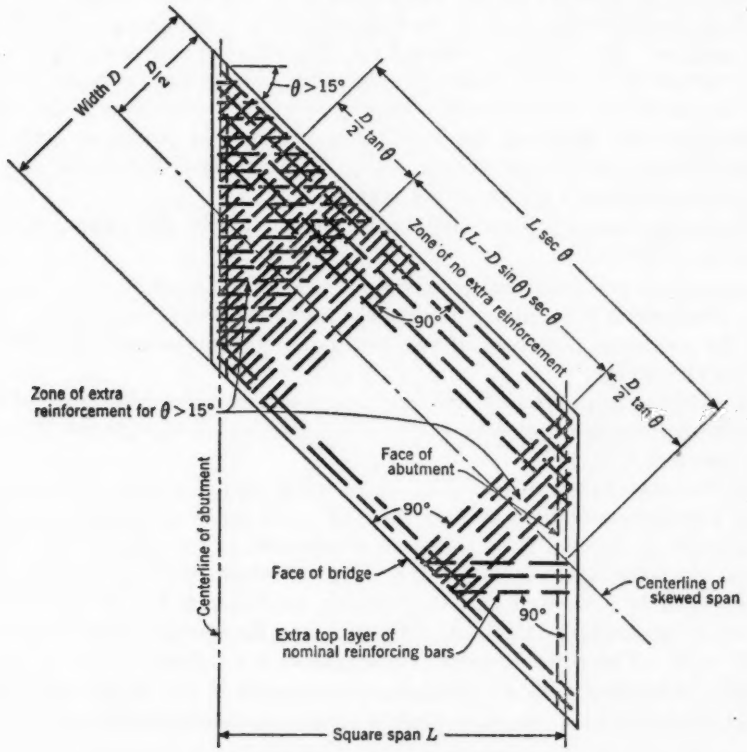

FIG. 17.—REINFORCEMENT PLAN

explains why many bridges are excessively large at the point where the arch meets the abutment.

Imagine, temporarily, that the entire skewed bridge is made of a series of ribs, and that the generatrix of the intrados and the extrados of each rib is a line perpendicular to the face of the bridge during the time that it is generating both the vault and the abutment. Such an assumption would not be objectionable to the stress analyst. The abutments would appear as a saw-toothed surface, as would the vault.

If the intrados is considered to be a cylindrical vault with the generatrix parallel to the z-axis (as in the paper), the analyst must always bear in mind that the maximum positive or negative moment—which happens to occur at a

certain point near the region of one face of the bridge—will not be the same at a point of symmetry near the other face of the bridge and on the line normal to the longitudinal axis of the bridge from that point. This is true, even for symmetrical conditions of loading. Consequently, if a skewed bridge is designed on an assumed fictitious equivalent square span equal to $L \sec \theta$, the reinforcement along the entire bridge must be proportioned in accordance with the maximum moments resulting from the several conditions of loading and temperature variations.

It can be proved by comparative analysis that, for a small angle θ, the design based on the equivalent span $L \sec \theta$ is sufficiently satisfactory; but, for larger skews and for relatively large ratios L/D (Fig. 17), the equivalent span $L \sec \theta$ is very practical in that it creates an extra resistance of the vault at the skewed ends by increasing the nominal steel (by the same criteria as those used in determining the so-called nominal reinforcement for ordinary structures). This additional steel is shown in Fig. 17. It strengthens the top of the vault against the cantilever action of the skewed ends.

To summarize: A bridge, such as the one treated in this paper, can be designed as follows:

1. Analyze an equivalent square span of length $L \sec \theta$.
2. Proportion the main reinforcement along the entire vault in accordance with the maximum positive and negative moments, and locate it parallel to the faces of the bridge.
3. If the arch has vertical abutments and if there is no continuity of curvature, enlarge the mass of concrete in the region where the abutment and the arch intersect.
4. Place the nominal steel at an angle of 90° with the faces of the bridge.
5. For a skew angle θ greater than 15°, and for $L \geqq D \sin \theta$, provide an extra layer of nominal steel along the abutments at an angle of 90° with the face of the abutment of the bridge. Fig. 17 shows that, in passing from a square span to a skewed one, the limiting condition is $\theta = 0$, the main reinforcement is parallel to the face of the arch, and the nominal steel is at an angle of 90° to it. Therefore no extra reinforcement is required.

All the other details so masterfully presented in the paper will give the bridge analyst a very complete picture of this interesting subject.

KUANG-HAN CHU,[23] JUN. ASCE.—Because it simplifies the analysis of skewed reinforced concrete rigid-frame and arch bridges, this is a highly valuable paper. However, there seems to be a need for some explanation for dropping certain terms. For instance, from Eq. 20, without dropping any term, the following equations are obtained: For the vault of a rigid frame, $\sin \theta = 0$ and $\cos \theta = 1$; therefore—

$$M_v = \epsilon \cos \theta \, M_s + [-(B - \epsilon A) u + (D + \epsilon A h)] \ldots\ldots (50)$$

For the legs of a rigid frame, $\cos \theta = 0$, $\sin \phi = 1$, and $u = \frac{L}{2}$; therefore—

$$M_v = \frac{-(B - \epsilon A)L}{2} - C \ldots\ldots\ldots\ldots\ldots\ldots (51)$$

[23] Structural Designer, Ammann & Whitney, New York, N. Y.

It can be seen that Eq. 21 is obtained from Eq. 50 by dropping the term in the bracket and Eq. 22*a* is obtained from Eq. 51 by dropping the term C. The author states that the term $(B - \epsilon A)$ is almost equal to zero for rigid frames. For the vault the value of u is small, and the term $(B - \epsilon A)u$ may be dropped. Nevertheless, neither the term $(D + \epsilon A h)$ nor the term C is equal to zero; $C = 0$ only when the loads are symmetrical. In dropping these terms, some justification seems to be needed.

Since Mr. Barron touches the problem of design, another question may be worthy of notice. It is the validity of using the torsion formula for homogeneous uncracked sections in computing torsional stresses in reinforced concrete sections. The writer has not been able to find any authoritative theoretical or experimental treatment on this subject.

HAROLD E. LEVENSON,[24] JUN. ASCE.—The method developed by the author for the analysis and design of rigid-frame and arch bridges may be extended to those which are curved in plan or composed of forms other than a simple rectangle or rhombus. Such structures are suitable solutions to highway intersection problems where the alinement of the upper roadway is curved. A structure, such as shown in Fig. 18(*a*), has a minimum deck area for given span

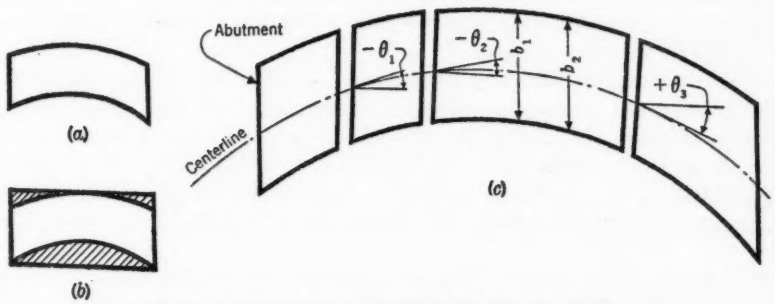

FIG. 18.—RIGID-FRAME OR ARCH BRIDGE, CURVED IN PLAN

and side clearances. It should be observed that, if a bridge of a parallelogram plan were used, for the same alinement and clearances, the additional darkened area of Fig. 18(*b*) would be required. In addition to its economy, this type of structure produces a pleasing architectural effect with its smoothly curved parapets.

For the purpose of analysis, the curved-in-plan structure may be cut with a series of sections parallel to the abutments as shown in Fig. 18(*c*). The normal at each section makes a skew angle θ with the center line. Thus a structure of "variable skew" is obtained. Also it should be noted that the width, b, is a variable when the roadway is of constant width.

Mr. Barron states that he has proved that the independence of the skew and rectangular effects for the cases of unsymmetrical structures include those of variable width. Accordingly, for a structure with a plan as in Fig. 18(*a*), using the author's simplified method of analysis, the equivalent rectangular structure,

[24] Harold E. Levenson Co., Development Engrs., Brooklyn, N. Y.

with the same span and with a width varying according to the width of the actual structure, is first analyzed. If the skew effects are to be determined, these may be determined by integration, either by the calculus or by the arithmetic summation of values for successive sections of length Δs.

The author has demonstrated that, for single-span structures of constant skew, the skew effects need not be considered in design when the skew angle is less than about 50°. For structures of variable skew of the type shown in Fig. 18(*a*), for radii of curvature of the alinement most commonly used in highway applications where the average skew is within the foregoing limit, the skew effects need not be considered in the design. The analysis should be based on the square span; the widths b will vary.

In the design phase of the work, it will be observed that, in addition to the stresses considered for the structure of constant skew, the curvature causes a lateral displacement of the line of action of redundant A from the centroids of the sections thus causing bending about a vertical axis at each section. However, for ordinary bridge sections the effect on the final design stress is less than 0.1%, due to the tremendous section modulus of the bridge for bending in this direction. Accordingly, this effect may ordinarily be disregarded.

The principle of the variable skew bridge may be extended to other cases.[25] In analyzing some of these structures, wedge-shaped segments may be taken, and the effects on the redundants may be evaluated in a way analogous to that used for structures of uniform skew.

The economy, simplicity of design, and architectural beauty of these new types of patented structures, especially the curved viaduct type, merit the attention of those responsible for the selection of highway bridge types, and currently several bridges with curved plans are being built. It is suggested that the testing of a series of curved-in-plan models to verify the theoretical considerations presented herein will prove a useful and fruitful investigation.

WALTER L. SCHWARTZ.[26]—The principal feature of this paper is the complete separation of the torsional effects from the rectangular effects, and the determination of the torsional effects, if required, as a product of a rectangular effect times an elastic or geometric constant.

This paper shows the care that must be exercised by researchers in comparing theoretical values with model tests data. Since the rectangular elastic system is shown to be independent of the torsional elastic system, it is not correct to conclude that the torsion theory is verified when the experimental data on test models agree with the rectangular results of the theoretical computations. Even a torsional analysis that is several hundreds per cent in error may yield correct results for the rectangular reactions and stresses. Test data previously correlated should be reviewed in this light.

From a practical standpoint the proof that a torsional analysis is usually of academic interest is a worthy step forward in the science of bridge building. The writer recalls an analogy from the author's personal correspondence which aptly describes this development:

[25] "The Rigid-Frame Bridge," by Arthur G. Hayden and Maurice Barron, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., 1950, p. 172.

[26] Brooklyn, N. Y.

"Suppose, in the first application of the three equations of static equilibrium to the simple truss [a simple two-panel Warren truss] one additional secondary truss equation was written for each of the five joints and all eight equations were then solved simultaneously. The analysis and design would be correct and model tests would prove the validity of the method, but only experts would be able to design this simple structure."

The solution of numerous simultaneous equations for skewed structures involving rectangular and torsional redundants, although much more obscured, represents similar self-imposed complexity.

LEO SOS[27].—Much has been written concerning the inconsistencies in computations when simultaneous equations are solved numerically and small differences are encountered. In comparing the results of the author's method with the published analysis and design of a skewed rigid-frame bridge[28] several inconsistencies were found and small differences were suspected as being the

TABLE 11.—SIMULTANEOUS EQUATIONS

No.	Equations	Vertical Unit Load at Point No.:			Earth pressure left	Temperature change
		6 R	8 R	10 R		
(1)	(2)	(3)	(4)	(5)	(6)	(7)
I	$+119.664\ R_x - 30.867\ R_z + 0.1341\ M_x - 0.5163\ M_y$	$= +19.72$	$= +37.57$	$= +48.15$	$= +13,039.0$	$= +1,215.0$
II	$+\ 11.561\ R_x - 43.735\ R_z + 1.418\ M_x - 0.2477\ M_y$	$= -\ 4.49$	$= -\ 8.61$	$= -11.27$	$= -\ 3,298.0$	$= -\ 356.0$
III	$+\ 0.0502\ R_x - 1.418\ R_z + 0.0664\ M_x + 0$	$= -\ 0.219$	$= -\ 0.424$	$= -\ 0.538$	$= -\ 146.9$	$= 0$
IV	$+\ 0.1934\ R_x - 0.2477\ R_z + 0 \quad - 0.0090\ M_y$	$= +\ 0.0057$	$= +\ 0.0078$	$= +\ 0.0024$	$= -\ 5.09$	$= 0$

TABLE 12.—COMPARISON OF FINAL VALUES

No.	Redundant	Vertical Unit Load at Point No.:			Earth pressure left	Temperature change
		6 R	8 R	10 R		
(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Moment M_y:					
1	Published =	−0.4581	+0.0658	+0.4296	+485.01	−1,477.8
2	Recomputed =	−0.49653720	+0.11116773	+0.39951280	+487.94191722	−1,475.33820977
	Moment M_x:					
3	Published =	−0.2218	−0.9908	−0.7040	+ 76.56	+1,466.0
4	Recomputed =	−0.19077984	−0.99691506	−0.67406015	+ 74.56316051	+1,463.19540918
	Reaction R_x					
5	Published =	+0.1515	+0.2660	+0.3642	+112.14	+ 69.2
6	Recomputed =	+0.15275882	+0.26542941	+0.36551424	+112.04037350	+ 69.22454545
	Reaction R_z					
7	Published =	+0.2021	+0.3838	+0.4988	+139.88	+ 20.0
8	Recomputed =	+0.20226988	+0.38402592	+0.49913904	+139.88560980	+ 20.00454282

cause. Therefore, the four simultaneous equations as published were recomputed to eight significant places and the results compared with published results which were computed only to four significant places.

Table 11 shows the four equations, and Table 12 compares the final values of the published and recomputed computation. Inconsistencies range almost to 100%.

[27] New York, N. Y.

[28] "The Rigid-Frame Bridge," by Arthur G. Hayden, John Wiley & Sons, Inc., New York, N. Y., 2d Ed., 1940, p. 154.

Every skewed structure analyzed by numerical solution of simultaneous equations will manifest these inconsistencies unless carried to extreme accuracy. Unless the designer has great experience there is no way of determining beforehand the number of significant digits to carry.

The author's method of analysis eliminates this troublesome problem and can be used with much more confidence. There is less chance for error and a great saving in design time.

LEIF ARUP.[29]—Since Mr. Rathbun presented his theory of skew arch analysis in 1924,[2] various simplifications of the design method have been suggested. The value of these suggestions, of course, depends on the validity of the original assumptions. In the writer's opinion Mr. Barron's paper is the final word based on Mr. Rathbun's original theory, which is the basis of all current theories on skew arch analysis.

Mr. Barron has set up a timesaving method of analyzing the structures, which has enabled him to arrive at conclusions and simplifications quickly and, in a way, that makes the problems easy to visualize. By setting up such a "pilot design" and by stating in a clear and comprehensive manner which factors can be neglected and which simplifications are justified, the task of future designers of skew bridges should be lightened. The design can be based on the properties of a square span, by using the prescribed adjustments and limitations.

Mr. Barron's pilot design involves a vastly reduced amount of work but yields identical results with those obtained from Mr. Rathbun's theory. The general derivation and extension of the elastic loads to include torsion, and the use of these and the special redundant system shown in the paper were first applied by the writer in 1928 to the design of Bridge No. 1, County Road 68, White Plains, N. Y. A paper containing the methods and a sample design was checked by the late A. Ostenfeld, in Copenhagen, who issued a statement of approval (1928). This paper, however, was never published.

Mr. Barron's conclusions can be accepted if Mr. Rathbun's theory is accepted as valid. The writer, however, concurs with Alfred L. Parmé,[30] Assoc. M. ASCE, and J. Polivka,[31] M. ASCE, in their conclusions that the principles from which Mr. Rathbun derived his basic equations are theoretically incorrect. Two objections which affect this theory, and consequently all work based on it, are as follows:

1. The method is bound to the use of sections not normal to the axis of the member for computing deflections. This is in contradiction to the theory of beams, and causes an error as explained subsequently in Example 1.

2. The theory of beams is applied to the analysis of members, the width of which is not small compared with the span. This is not in accord with the definition of a beam. The consequences of this procedure will be demonstrated subsequently in Example 2.

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[29] Cons. Engr., New York, N. Y.

[30] Discussion by Alfred L. Parmé of "Simplified Analysis of Skewed Reinforced Concrete Frames and Arches," by Richard M. Hodges, *Transactions*, ASCE, Vol. 109, 1944, pp. 954–960.

[31] Discussion by J. Polivka of "Simplified Analysis of Skewed Reinforced Concrete Frames and Arches," by Richard M. Hodges, *ibid.*, pp. 967–968.

Example 1.—Consider a cantilever beam (see Fig. 19), for which the length l is large in comparison with the breadth b. The problem is to find the vertical deflection of the point of application of a load P. By the usual method of analyzing sections normal to the axis of the beam, the result will be

$$\Delta_N = \frac{P\,l^3}{3\,E\,I} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (52)$$

in which $I = \frac{b\,t^3}{12}$. By using the skew sections—that is, taking sections forming an angle θ with the normal to the axis, and by combining the flexural and tor-

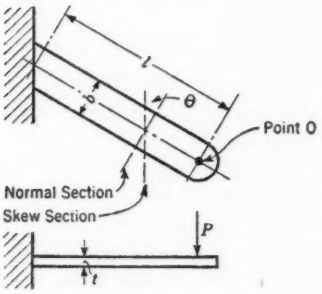


FIG. 19

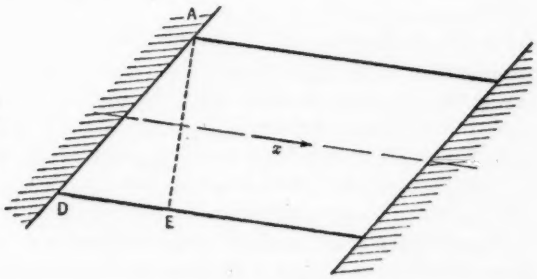


FIG. 20

sional deformations in these sections (as is done in the paper)—it can be shown that the resultant deflection by this method is

$$\Delta_s = \frac{P\,l^3}{3\,E\,I} \times K_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53)$$

in which

$$K_1 = \cos^2\theta\,[\cos^2\theta' + \sin^2\theta'\,\tfrac{3}{5}\,(1-\mu)] = f\,(\theta', \mu) \ldots\ldots\ldots (54)$$

This means that the deflection is itself a function of the angle θ' and of Poisson's ratio, μ. How large is the difference in the results obtained by Eqs. 52 and 53?

Solving Eq. 54 with $\mu = 0.15$, a few K_1-values for selected skew angles are as follows:

θ'	K_1
0°	1.00
20°	0.84
40°	0.51
60°	0.19

In other words, for $\theta' = 60°$, Eq. 53 becomes

$$\Delta_s = \frac{P\,l^3}{3\,E\,I} \times 0.19 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (55)$$

which is about one fifth the value obtained by Eq. 52.

Since redundants in statically indeterminate structures depend on ratios between deflections rather than on the deflections themselves (except volume changes and foundation yield), the possibility might exist that K_1 enters as a

constant factor in all deflections and thus cancels out. That this is not the case can be shown by investigating other deflections as, for example, the angular rotation ψ at point O around the z-axis, due to the load P. By using square sections the result will be

$$\psi = \frac{P\,l^2}{2\,E\,I}\cos\theta \qquad (56)$$

and by using the skew sections

$$\psi = \frac{P\,l^2}{2\,E\,I}\cos\theta \times K_2 \qquad (57)$$

in which

$$K_2 = \cos^2\theta \qquad (58)$$

Since $K_1 \neq K_2$ the ratio between deflections differs whether square or skew sections are used in the analysis. That such factors as K_1 and K_2 are not constants can also be seen from the fact that they are functions of θ'. The angle θ' is the angle between the sections and the line normal to the axis. In an arch this angle varies from point to point along the axis and, in a rigid frame, θ' closely equals the skew angle in the vault; it is O° in the abutments. Thus contributions to the deflections due to abutments are independent of Poisson's ratio μ and the skew angle θ; but contributions from the vault are functions of these quantities. Results obtained in this manner must obviously be arbitrary and incorrect.

Example 2.—Mr. Barron reaches the conclusion expressed in Eq. 21 which states that, for all vertical loads, $M_y = \epsilon\, M_x$ ($\cos\phi = 1$ for a flat topped frame). This means that the principal moments are always acting on planes normal to the faces of the bridge. Therefore, no torsional moments would be acting on these planes and consequently no torsional shears. Applying the theory of plates, the basic expression for torsional shears is

$$v_{xy} = -\frac{E}{1+\mu}\,\zeta\,\frac{\partial^2 w}{\partial x\,\partial y} \qquad (59)$$

in which μ is Poisson's ratio; ζ is the distance from the neutral plane; and w is the deflection at any point of the neutral plane, with the rectangular coordinates (x, y).

Choosing the x-axis parallel to the faces of the bridge, the torsional shear v_{xy} should be equal to zero. This is possible only if $\frac{\partial^2 w}{\partial x\,\partial y} \doteq 0$. In the general case the deflection w is a function of both coordinates (x, y). That is, $w = f_1(x, y)$ and

$$\frac{\partial^2 w}{\partial x\,\partial y} = \frac{\partial\left(\frac{\partial w}{\partial x}\right)}{\partial y} \qquad (60)$$

in which $\frac{\partial w}{\partial x}$ is the slope of the deflected neutral plane in the direction of the x-axis, or parallel to the face of the bridge. Now $\frac{\partial\left(\frac{\partial w}{\partial x}\right)}{\partial y}$ can be equal to zero

for all points only if $\frac{\partial w}{\partial x} = f_2(x)$—that is, only if $\frac{\partial w}{\partial x}$ is a function of x exclusively, which means that the slope of the deflection surface in the direction of the skew span is constant in a given section normal to the faces of the bridge. The inconsistency of this requirement becomes evident by considering a skew slab fixed at the supports as in Fig. 20. This case corresponds to a flat-topped, skewed, rigid frame, with infinitely stiff legs, for which Eq. 21 is supposed to be exact, as shown originally by Mr. Rathbun.[32]

It may now be seen from Fig. 20 that, at point A, as well as all along the fixed edge A–D, the slope $\frac{\partial w}{\partial x}$ is equal to zero. This makes $\frac{\partial w}{\partial x} = 0$ at all points inside the triangle ADE, which again means that this part of the structure does not deflect as the result of external vertical loading. That this conclusion is incorrect for wide structures seems obvious. It may be approximately true for very narrow slabs, where the width is negligible compared to the span—in other words, when the slabs are not slabs but beams.

From these considerations the writer concludes that Eq. 21, especially for wide structures, is not correct.

Since the elastic theory of beam analysis defines a beam as a member with two dimensions (depth and width) which are small compared with the third (span), it follows that this definition does not hold true for slabs, whether square or skew. It happens, however, that the method of analyzing beams satisfies the differential equation and all boundary conditions for a rectangular slab, supported on two opposite edges, provided that Poisson's ratio is assumed to be zero, which is a minor discrepancy. For skew slabs the boundary condition that the reaction along the free edges should be equal to zero is not satisfied, but gives antisymmetrically distributed reactions for a symmetrical loading. This can be seen by an inspection of the differential equation for the elastic slabs transformed into skew coordinates by Henry Favre.[33] Methods for approximating a solution that satisfies this condition are rather involved and cannot be given here, but the fact that this contradictory reaction is antisymmetrically distributed already indicates that its effect decreases from the free edge toward the center of the slab, and that the magnitudes of the moments near midspan are less affected than their directions. These conculsions are also verified by the many numerical examples solved by V. P. Jensen,[34] M. ASCE.

Since the writer finds that the moment trajectories for vertical loads do not follow lines parallel to the faces of the bridge (as Eq. 21 would indicate), but vary considerably in direction, the reinforcing steel for practical reasons cannot be placed to coincide with the moment trajectories. A method of designing the steel under these circumstances is needed. It has been indicated[35,34] that the practice of multiplying the area of the steel required in the direction of the moment trajectories by the $\sec^2$ of the angle between these and the direction of

[32] "Analysis of the Stresses in the Ring of a Concrete Skew Arch," by J. Charles Rathbun, *Transactions*, ASCE, Vol. LXXXVII, 1924, p. 624, Eq. 16.

[33] "Contribution à l'étude des plaques obliques," by Henry Favre, *Schweizerische Bauzeitung*, Vol. 120, 1942, pp. 35–60.

[34] "Analyses of Skew Slabs," by V. P. Jensen, *Engineering Experiment Station Bulletin Series No. 332*, Univ. of Illinois, Urbana, Ill., 1941.

[35] "Laboratory Research on Concrete Bridge Floors," by F. E. Richart, *Transactions*, ASCE, Vol. 114, 1949, p. 980.

the steel is not adequate. To explain this statement, consider the reinforcement designed in the direction of the moment trajectories (Fig. 21(*a*)), and for simplicity let this coincide with the direction of the square span. A small element (Fig. 21(*b*)) is cut out and it is assumed that a uniform tension p is acting along the two edges of the element which are normal to the main steel. The tension in each bar will then be $P = p\,d_R = A_{sR}\,f_s$ for which it is designed. Now the main bars placed in the direction of p are substituted by the same number of bars placed on the skew (Fig. 21(*c*)). These will form an angle θ with the direction of the tension p.

It is now obvious that to produce the same component in the direction of p, the area must be increased sec-θ times and, since the spacing is multiplied by $\cos\theta$, the areas per unit width measured normal to the bars should be multiplied

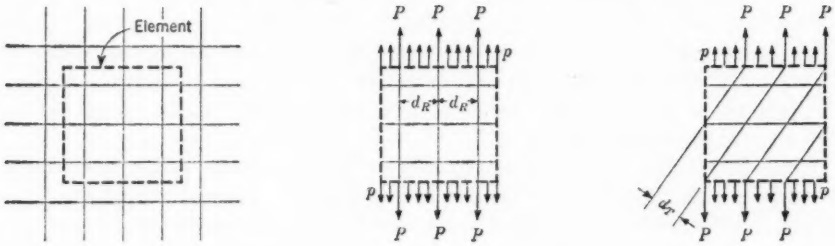


FIG. 21

by $\sec^2\theta$. This procedure would be identical to the "turning process" described by Mr. Barron. However, the reinforcement in the element is not stable, and the main bars tend to rotate in counterclockwise direction for the given loading condition. This is prevented by stresses in the concrete, which again introduce additional tension in the steel. The direction of these concrete stresses can be found by determining the direction in which the maximum and minimum unit elongations of a fiber in the plane of the element occur, for an infinitesimal rotation of the main bars. These directions can be shown to coincide with the bisectors $\xi-\xi$ and $\eta-\eta$ of the angles at the intersection of the bars. Assuming that concrete can act in compression only, this force will act in the direction of $\xi-\xi$, Fig. 22. At the edge of the element the forces must be in equilibrium; P, acting as an external load on the element, must balance the steel tension T and the compressive force C in the concrete (see Fig. 23). By projecting on the axis, $\eta-\eta$, C is eliminated, and

$$T\cos\left(45^\circ+\frac{\theta}{2}\right)-P\cos\left(45^\circ-\frac{\theta}{2}\right)=0 \qquad (61)$$

which leads to the formula:

$$\frac{A_{sT}}{d_T}=\frac{A_{sR}}{d_R}\times\sec^2\theta\,(1+\sin\theta) \qquad (62)$$

If the moment trajectories fall between the square and the skew directions and

they form an angle γ with the square direction, a similar consideration will give

$$\frac{A_{sT}}{d_T} = \frac{A_{sR}}{d_R} \times \sec^2\theta\,(1 + \sin\theta - \tan\gamma\sec\theta)\cos^2\gamma \ldots\ldots\ldots (63)$$

In Eqs. 61, 62, and 63 $\frac{A_{sT}}{d_T}$ is the required reinforcing steel area per unit width, measured normal to the direction of steel; $\frac{A_{sR}}{d_R}$ is the fictitious reinforcing steel area per unit width computed on the assumption that the direction of steel coincides with the moment trajectories; and γ is the angle between square span

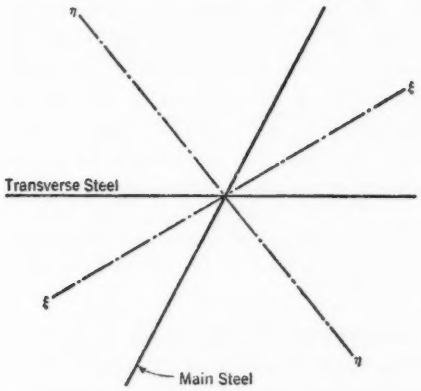


FIG. 22

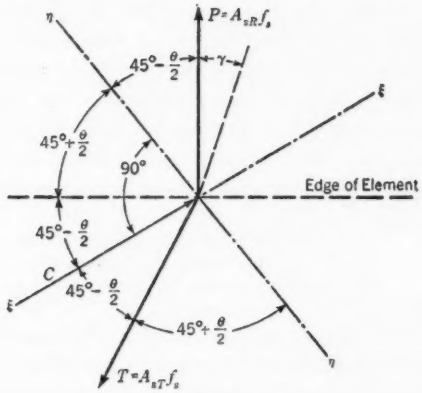


FIG. 23

direction and moment trajectories. For skew bridges, the safest procedure seems to be to use $\gamma = 0$ (moments assumed to be acting on the square). A more exact analysis based on the theory of plates may permit reducing the required amount of steel when $\gamma > 0$.

Summary.—In conclusion, the writer wishes to state that, in his opinion, Mr. Barron has drawn all possible conclusions from Mr. Rathbun's work in an admirable manner. Since the introduction of the theory of plates and shells,[33,34,36] however, Mr. Rathbun's theory has become obsolete. Future investigations on skew arches should be based on modern research on plates and shells.

Acknowledgment.—The mathematical computations that support this discussion were made by Walter M. Zalite.

MAURICE BARRON,[37] M. ASCE.—The number of discussions and the constructive criticism they contain are encouraging to the writer. One type of discussion is lacking, however. The author hoped that engineers and designers would review their old designs based upon previously published methods[2,3,4,8]

[36] "Die strenge Lösungdes, Problems der schiefen Platte," by P. Lardy, *Schweizerische Bauzeitung*, 1949, p. 207.

[37] Cons. Engr., Farkas & Barron, New York, N. Y.

[3] "The Rigid-Frame Bridge," by Arthur G. Hayden, John Wiley & Sons, Inc., New York, N. Y., 1931.

[8] "An Analysis of Multiple-Skew Arches on Elastic Piers," by J. Charles Rathbun, *Transactions*, ASCE, Vol. 98, 1933, p. 1.

and compare the results with those obtained by this practical office procedure. It may be assumed that several designers have made these comparisons and, finding them in agreement, have not reported the results. Some discussers have idealized the problem by application to very simple structures. Properly applied, the methods proposed in this paper yield the same results.

In any comparison of an analytical solution with test data (or with another analytical method) careful consideration must be given to the interpretation of results. If no correlation is shown, further tests must be applied to determine which is incorrect. Furthermore, even if exact correlation is shown, conclusions may still be incorrect because both results may be wrong to the same degree.

Mr. Nettleton compares his application (Method A) of the writer's method, and his application of his own method (Method B) to an idealized frame of uniform moment of inertia which consists of vertical legs and a horizontal vault with a uniform load applied. The results obtained by Method A are not in agreement with the results obtained for the same structure by the writer. The discrepancies are due to Mr. Nettleton's misinterpretation of the turning process. This process is not reversible. The statement by Mr. Nettleton—

> "* * * 'Turning' the moments instead of the required areas of reinforcing results in the same requirements for reinforcing, and is done here to facilitate a comparison between the two methods."

—contains a false premise. The reader is advised to think in terms of a homogeneous material, such as aluminum, rather than in terms of reinforced concrete. In the conventional design of reinforced concrete, unit shearing stresses in the concrete cannot be combined vectorially with the unit compressive stresses in the concrete. However, when a reinforcing bar is placed in a plane other than the one in which the bending moments and thrusts are computed, the tension (or compression) in the rod can be resolved into two components as shown in Fig. 8. It should be noted that turning the reinforcing does not imply that the concrete fibers are also turned. Turning the moments as Mr. Nettleton has done usually results in incorrect answers.

For example,[38] referring to a design section for the vault of a single-span rigid frame and using the notation of the paper, let: $M_s = 33.3$ kip-ft; $T_v = 11.2$ kips; $t = 25$ in.; $d = 23$ in.; $a = 10.5$ in.; $\theta = 47°50'$; $\epsilon = 1.1$; $\epsilon^2 = 1.21$; $\sec^2\theta = (1 + \epsilon^2) = 2.21$; $f_s = 18$ kips per sq in.; $f_c = 1$ kip per sq in.; $n = 10$; and $K_b = 157.3$. Also: $e = \frac{M_s}{T_v} = \frac{33.3 \times 12}{11.2} = 35.6$ in.; $e' = e + a = 35.6 + 10.5 = 46.1$ in.; $M_s = e' T_v = 46.1 \times 11.2 = 516$ in.-kips; and $K = \frac{M_s}{12 \times d^2} = \frac{516}{12 \times 23^2}\ (\times 1{,}000) = 81$. Since $K\ (= 81)$ is less than $K_b\ (= 157.3)$, f_c is less than 1,000 lb per sq in. Consequently, $A_{sR} = \frac{M_s}{f_s j d} - \frac{T_v}{f_s} = \frac{516}{0.9 \times 18 \times 23} - \frac{11.2}{18} = 1.42 - 0.62 = 0.80$ sq in.

[38] "The Rigid-Frame Bridge," by Arthur G. Hayden and Maurice Barron, John Wiley & Sons, Inc., New York, N. Y., 3d Ed., 1950, point 8, p. 159.

If the steel is turned into its most practical direction (parallel to the face), $A_{sT} = A_{sR}(1 + \epsilon^2) = 0.80 \times 2.21 = 1.76$ sq in. Turning the steel, however, does not change the unit compressive stress or the total compression. If the moments and thrusts are turned by a transformation,[4] K becomes (numerically) $81 \times 2.21 = 179.5$, which is greater than 157.3. Compression steel and additional tension steel is indicated. However, the transformation of moments cannot be justified because a unit shearing stress and a unit compressive stress have, in effect, been added vectorially. This example illustrates one important key to the design problem. All total stresses may be added vectorially but not all unit stresses. Mr. Nettleton's proposal for transverse reinforcement, using Mr. Jensen's formula,[11] does not include the effects of earth pressure or volume changes, which are so important in this type of structure. Furthermore, as far as the writer is aware, no exact analysis for skewed slabs by difference equations has been developed for variable moment of inertia, with edges stiffened by parapet walls, or with unrestricted deck curvature such as encountered in the structures under consideration. Mr. Nettleton contends that:

> "The design of the transverse reinforcement should be controlled by the transverse moments, and not by the torsional moments, because the torsional moments are generally so small that no transverse reinforcing is required."

It should be noted that the torsion moment, M_v, for the vault of a 45° skewed rigid-frame bridge is equal to M_x and for larger skews M_v is even greater than M_x. The reinforcement pattern should be designed to provide for all moments and shears due to all loads. Review of Mr. Nettleton's discussion, however, has revealed one interesting point in regard to the value of M_x for vertically applied uniform loads. Referring to Table 5 under dead load and to Eq. 23*d*: By evaluating the right-hand side of Eq. 23*d*, it can be shown that (at least, for rigid-frame bridges) D is equal to ϵ times the average of the two fixed end moments as usually determined. This statement is also valid for variable I and for any concentrated vertical load. Therefore, the evaluation of the foundation restraint M_x becomes very simple if tables[39,40,41] for fixed end moments are used.

From Mr. Harder's discussion it appears at times that he is in complete agreement with the proposed method and at other times diametrically opposed to it. Comparison of the results of his design with the results of this method reveals that both are in agreement. The easiest way to test the proposed method is to make these comparisons. To the writer's knowledge, no skewed structure was ever analyzed or designed using sections at right angles to the face. Hundreds of skew structures, designed and investigated by the writer, clearly show the necessity for a practical design procedure. No structure, carefully designed by qualified engineers using accepted published methods, has been endangered. The elastic theory, as applied to these structures, has been proved satisfactory by these many successful bridges, adequately performing

[39] "Handbook of Frame Constants," Portland Cement Association, Chicago, Ill.

[40] "Beam Factors and Moment Coefficients for Unsymmetrical Members with Parabolic Haunches," Portland Cement Association, Chicago, Ill.

[41] "Continuous Concrete Bridges," Portland Cement Association, Chicago, Ill., 2d Ed., 1941.

their intended function. Failures and structures manifesting distress are clearly the result of inadequate design, detailing, or construction procedure. It is imperative that the structural action on which the design is predicated be realized. The foundations must fulfil their intended function, and the details must insure the structural action intended.

Mr. Harder, criticizing the design of the transverse reinforcement, states that an error is made in this paper, as well as in Mr. Hodges' paper.[4] This statement ignores the fact that all loadings and stresses are found for a 12-in. width of structure. Thus, the moments are given in kip-feet per foot of width and the shears in kips per foot of width. The resulting steel areas are for 12 in. and not 1 in. as Mr. Harder contends. This result is evident if the contention is checked against a numerical example. Mr. Harder's solution results in concrete sections containing as much as 50% steel reinforcing—an absurdity. The skew width of structure, b, is a linear parameter in each equation and in each step of the derivations. This fact is used to advantage, permitting the structure to be designed for a width of 1 ft. It also permits total effects to be determined by multiplying the values for a unit width by b, the skew width of the structure.

Mr. Harder requests clarification on several points. Concerning the relative importance of M_u, he is confusing M_u with M_y and M_z, which are always included in the method proposed in this paper. It should be noted that the moments M_z and M_y cause the major shift of R_y and R_z. Neglecting M_u in the analysis part is less significant than the omission of the deflection due to shear, and in the design part it has been shown several times that M_u causes insignificant stresses. Mr. Harder states that:

> "* * * Mr. Rathbun follows the lines that this writer [Mr. Harder] prefers—namely, transposing the reactions, as computed, into components parallel to the skew center line and at right angles thereto. * * * In the design of sections, Mr. Barron uses an oversimplified method which * * * virtually assumes that moment M_L and thrust T_L are acting on the diagonal section parallel to the abutments. * * * What about the last diagonal section at the face of the abutment? How can one justify the 'turning process' there?"

The method Mr. Harder favors and the proposed method yield the same results; but the writer determines the end product mathematically whereas Mr. Harder substitutes numerical values halfway through the operation of finding design stresses. Premature substitution of numerical values results in loss of clarity, a loss in accuracy, and a great loss in time.

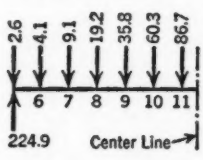
Mr. Di Genova presents an approximate solution based on using the skewed span. In seeking simplification over a period of years, a solution similar to that of Mr. Di Genova was explored along with several other simplifications and approximate methods. A right-arch analysis using the skewed span was abandoned many years ago because it was found that this approximation was incorrect for stresses due to earth pressure, volume change, and live loads. It is fairly accurate for symmetrical vertical loads on symmetrical, single-span, rigid-frame bridges. For small skews the error is not serious. For larger skews the error is on the dangerous side when the right-arch analysis is applied

to the vault of rigid frames. For the legs of rigid frames the error is on the safe side but results in wastefulness. An interesting point is raised by Mr. Di Genova in his discussion concerning reinforcing patterns. The pattern shown in Fig. 17 requires the cutting and placing of many short individual bars. Furthermore, when the vault is arched and the thickness varies, the difficulty in tying the intrados and extrados reinforcement together presents a problem because there is no uniformity in the depth of stirrup. This condition results from placing transverse steel perpendicular to the face of the bridge.

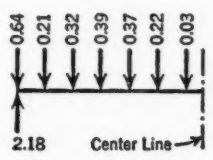
Several designers, including Mr. Chu, have found difficulty in properly expanding Eq. 20 and combining the terms. It should be carefully noted that each type of loading (vertical loads, horizontal loads, volume change) requires a different grouping for the vault and for the legs in order to separate primary from secondary effects.

Mr. Chu requests further explanation and justification for dropping out terms such as the terms within the bracket of Eq. 20. In order to show that $(B - \epsilon A)u$ may be neglected, Eq. 23*b* is used. Of course, for an idealized frame with flat vault, Eq. 23*b* reduces to zero because Δ_b is exactly zero (for

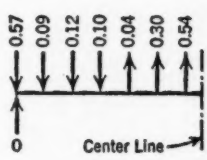
TABLE 13.—RECTANGULAR AND TORSIONAL ELASTIC WEIGHTS AND MOMENTS FOR UNIT VERTICAL INFLUENCE LOAD[a] FOR A SINGLE-SPAN SKEWED BRIDGE



Elastic Weight = y/I



Elastic Weight = $\epsilon r x \sin \phi \cos \phi / I$

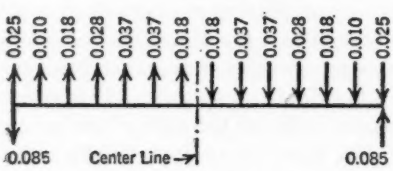


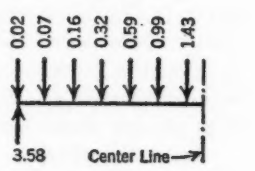
Elastic Weight = $\epsilon r u \cos \phi / I$

Point	(a) Deformation Δ_a (Rectangular) Elastic weight	Shear	Elastic moment	(b) Deformation Δ_a (Torsional) Elastic weight	Shear	Elastic moment	(c) Deformation Δ_b Elastic weight	Shear	Elastic moment
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	0.03		0	0		0	0		
2	2.32		0	0		0	0		
3	2.51		0	0		0	0		
4	2.22		0	0		0	0		
5	2.63	215.2	0	0.64	1.54	0	+0.57	0.57	
6	4.05	211.1	215.2	0.21	1.33	1.54	+0.09	0.66	0.57
7	9.10	202.0	426.3	0.32	1.01	2.87	+0.12	0.78	1.23
8	19.20	182.8	628.4	0.39	0.62	3.88	+0.10	0.88	2.01
9	35.80	147.0	811.2	0.37	0.25	4.50	−0.04	0.84	2.89
10	60.30	86.7	958.2	0.22	0.03	4.75	−0.30	0.54	3.73
11	86.70		1,044.9	0.03		4.78	−0.54		4.27
½Σ	224.90			2.18			0		

[a] Constants (ϵ, r, Δ_a, Δ_x, etc.) which are common factors are not included in the numerical values.

TABLE 13.—(CONTINUED[a])





Point	Elastic weight	Arm	Static moment at center line	Shear	Elastic moment	Elastic weight	Shear	Elastic moment
(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
1	0				0	0		0
2	0		1.10/13		0	0		0
3	0		0.085		0	0		0
4	0				0	0		0
5L	−0.025	13	0.32		0	0.02		0
5R	+0.025			−0.060			3.56	
6L	−0.010	11	0.11		−0.060	0.07		3.56
6R	+0.010			−0.050			3.49	
7L	−0.018	9	0.16		−0.110	0.16		7.05
7R	+0.018			−0.032			3.33	
8L	−0.028	7	0.20		−0.142	0.32		10.38
8R	+0.028			−0.004			3.01	
9L	−0.037	5	0.18		−0.146	0.59		13.39
9R	+0.037			+0.033			2.42	
10L	−0.037	3	0.11		−0.113	0.99		15.81
10R	+0.037			+0.070			1.43	
11L	−0.018	1	0.02		−0.043	1.43		17.24
11R	+0.018			+0.088				
½Σ			1.10			3.58		

vertical loads). Where the vault has a rise,[6] such as the usual rigid-frame bridge and in arches, Δ_b is not zero and u for the vault is not zero. Using torsional elastic weights it has been shown (unabridged paper[5a] filed for reference in Engineering Societies Library) that, even for the full center arch, Δ_b is relatively small, and the resulting correction ($= B - \epsilon A$) in the cross shear may be neglected. Examination of the elastic loads comparing y/I with $\epsilon r u \cos \phi / I$ and the elastic moments (see Table 13), indicates the relative value of ϵA and $(B - \epsilon A)$.

For symmetrical vertical loads on symmetrical structures (skewed rigid frames and arches), C is equal to zero, because Δ_c is equal to zero. Dissymmetry of load and usually dissymmetry of structure can be evaluated as secondary effects using the method shown in Table 13.

The following demonstration is not an exhaustive proof (except for the idealized frame), but it can be used to show that $(D + \epsilon A h) \cos \phi$ is never

[6] "The Rigid-Frame Bridge," by Arthur G. Hayden and Maurice Barron, John Wiley & Sons, Inc., New York, N. Y., 1950.

[5a] 29 W. 39th St., New York, 18, N. Y.

more than of secondary order. For an idealized frame, this term is zero for the legs ($\cos\phi = 0$) and equals ($D + \epsilon A h$) for the vault ($\cos\phi = 1$). Using Eqs. 23*a* and 23*d*:

$$(D + \epsilon A h) = -\epsilon Q_3 A - Q_7 \Delta_d + \epsilon A h \quad \ldots\ldots\ldots\ldots (64)$$

and for an idealized frame the following expressions result from substitution of values in Table 2: Eq. 14*c* becomes—

$$Q_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (65a)$$

Eq. 14*g* becomes—

$$Q_7 = \frac{1}{\delta_{dd}} = \frac{1}{r \sum \cos^2 \frac{\phi}{I}} = \frac{1}{r \sum_0^L \frac{1}{I}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (65b)$$

Eq. 9 becomes—

$$h = \frac{\sum u_0 \cos \frac{\phi}{I}}{\sum \cos^2 \frac{\phi}{I}} = y \quad \ldots\ldots\ldots\ldots\ldots\ldots (65c)$$

Eqs. 6*a* and 23*a* yield—

$$A = -Q_4 \Delta_{aR} = \frac{\sum \frac{M_{0z}\, y}{I}}{\sum \frac{y^2}{I}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (65d)$$

and Eqs. 6*b*, 8, and 19 yield—

$$\Delta_d = r \sum \left(\epsilon \cos\phi\, M_{0z} \cos \frac{\phi}{I} \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (65e)$$

Substituting Eqs. 65 into Eq. 64:

$$(D + \epsilon A h) = 0 - \frac{\epsilon r \sum M_{0z} \cos^2 \frac{\phi}{I}}{r \sum_0^L \frac{1}{I}} + \epsilon \frac{\sum \frac{M_{0z}\, h}{I}}{\sum \frac{y^2}{I}} h = -\frac{\epsilon \sum \frac{M_{0z}}{I}}{\sum_0^L \frac{1}{I}}$$

$$+ \epsilon \frac{\sum \frac{M_{0z}\, h^2}{I}}{\sum \frac{y^2}{I}} = \epsilon \left[\frac{\sum \frac{M_{0z}}{I}}{\sum \frac{y^2}{h^2 I}} - \sum \frac{\frac{M_{0z}}{I}}{\sum_0^L \frac{1}{I}} \right] \quad \ldots (66)$$

The numerators of the two fractions in Eq. 66 are identical. For an idealized frame the denominators are almost identical. For several frames and for two full-centered arches, the computations show that ($D + \epsilon A h$) is of secondary order.

Therefore, since the three terms in the brackets of Eq. 20 are of secondary order, their combination is of even less order. This has been confirmed by actual application to several structures.

Table 13 shows the conjugate structure (for a single-span, skewed bridge) loaded with elastic weights which are used to determine the redundant reactions resulting from a unit vertical influence load. Table 13(*a*) shows how to find the

moments due to the rectangular elastic weights; and Table 13(*b*) represents the same conjugate structure loaded with torsional elastic weights which are used to evaluate $(B - \epsilon A)$. By the law of superposition, these effects may be added together for the exact analysis. Comparison of the moments, however, shows that the additional effects due to torsion are negligible. It should be further observed that the torsional effects in Eq. 12 are compensating to a certain extent because their influence is reflected on both sides of the equation and to the same degree. Table 13(*c*) shows how to determine Δ_c for unit vertical influence load. Eq. 11*c* (or Eq. 16*c*) yields the redundant C from Δ_c and an elastic constant Q_6 (Eq. 14*f*). Table 13(*d*) shows the same procedure for Δ_d. A critical examination of each term supplies the best way to evaluate the slight errors introduced by dropping terms. Emphasis is placed on the fact that Δ_{aT}, Δ_b, Δ_c, and Δ_d, for each loading, may be used individually to evaluate the terms considered as secondary effects. Also, each major effect may be written as an elastic or geometric constant times a rectangular stress (conclusion 6).

Mr. Chu would have clarified this problem more had he made a numerical comparison of the proposed methods with actual designs in his office. The torsion formula as used in this paper and in many previous papers[2,4,42] has been justified by numerous tests.[43,44,16] Until additional research justifies a more refined formula, the writer recommends the torsion formula used (see Fig. 13). This recommendation is justified by observations at the University of Toronto[5] (Toronto, Ont., Canada), and the recorded experience of Walter C. Boyer[45] and Gordon B. Fisher,[42] Juniors, ASCE. Several other references are available.[46] Many universities have studied and tested skewed structures, and the results may be found in theses filed at such university libraries as The Johns Hopkins University[42,45] (Baltimore, Md.), Rensselaer Polytechnic Institute (Troy, N. Y.), Massachusetts Institute of Technology (Cambridge, Mass.), and the University of Utah (Salt Lake City, Utah).

Mr. Levinson's contribution broaches several very useful and fascinating subjects. The proposed analysis shows that the rectangular system is independent of the torsional elastic system and that each equation in the analysis may be written with a function of the skew angle as a parameter. It then becomes obvious that one rectangular structure may be considered as the parent of an entire family of bridges, each bridge having a different skew. A typical cross section becomes a generating element. Furthermore, if properly manipulated, this typical element will generate a curved-in-plan structure, the elements of which have family characteristics, easily analyzed and designed.

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[42] "An Experimental Study of the Reactions of a Two-Span Skewed Rigid Frame Bridge," by Gordon B. Fisher, thesis presented to The Johns Hopkins University, at Baltimore, Md., in partial fulfilment of the requirements for a graduate school degree in 1948.

[43] "Skew Arch Reactions Measured by Reciprocal Method," by George E. Beggs, *Engineering News-Record*, July 21, 1927, pp. 106–107.

[44] "The Use of Models in the Solution of Indeterminate Structures," by George E. Beggs, *Journal of the Franklin Institute*, March, 1927, p. 375.

[5] *Bulletin No. 3*, Faculty of Applied Science and Eng., School of Eng. Research, Univ. of Toronto, Toronto, Ont., Canada.

[45] "Effect of Skew Angle on Rigid-Frame Reactions," by Walter C. Boyer, *Proceedings-Separate No. 32*, ASCE, September, 1950.

[46] "Practical Design of Solid-Barrel Reinforced Concrete Skew Structures," by Bernard L. Weiner, *Proceedings-Separate No. 39*, ASCE, October, 1950.

The horizontal and vertical geometry required for these new structures is much simpler than that required for the fitting of a parallelogram structure to a curved roadway. When the upper roadway is curved, the faces of the bridge may be concentric, resulting in a uniform width of upper sidewalk. When the lower roadway is curved, the horizontal clearance is made constant by providing

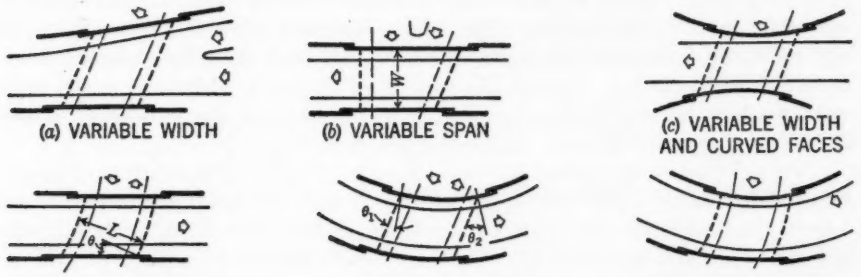
(a) VARIABLE WIDTH
(b) VARIABLE SPAN
(c) VARIABLE WIDTH AND CURVED FACES
(d) VARIABLE SKEW AND CURVED SUPPORTS
(e) VARIABLE SKEW AND CURVED FACES
(f) VARIABLE SKEW WITH CURVED FACES AND SUPPORTS

FIG. 24.—REINFORCED CONCRETE, SINGLE-SPAN, BARREL ARCHES AND RIGID-FRAME BRIDGES

abutments or legs concentric with the curvature. Figs. 24 and 25 show application of these principles and other possible combinations. In a manner similar to that shown by Mr. Levinson, Fig. 25(*e*) shows how the skew angle varies from θ_1 at the left abutment to θ_2 at the right abutment. It is readily seen that each element has a different skew angle.

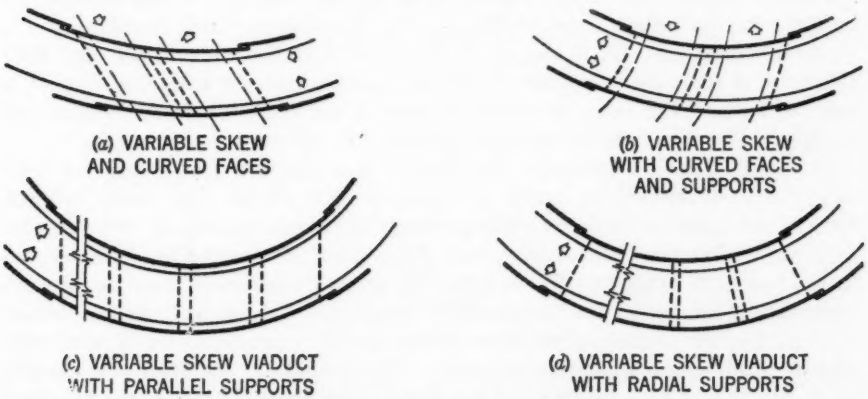
(a) VARIABLE SKEW AND CURVED FACES
(b) VARIABLE SKEW WITH CURVED FACES AND SUPPORTS
(c) VARIABLE SKEW VIADUCT WITH PARALLEL SUPPORTS
(d) VARIABLE SKEW VIADUCT WITH RADIAL SUPPORTS

FIG. 25.—REINFORCED CONCRETE, MULTIPLE-SPAN BARREL ARCHES AND RIGID-FRAME BRIDGES

The curved viaduct, consisting of a series of solid barrel rigid frames, may extend for an unlimited distance as a sequence of curved-in-plan and straight multispan rigid frames. This will result in an economical and aesthetic structure using a minimum of vertical depth. As much as 18 in. may be saved in the height of construction from the lower roadway surface to the upper roadway surface. The approach grades are much easier, and no separate wearing course or fill is required for these structures. Proper superelevation

may be incorporated in the structural vault. In Fig. 18(*b*) Mr. Levinson shows the deck area that can be saved by using curved-in-plan construction. This saving amounts to as much as 30% on some actual designs for currently proposed projects. Properly designed forms, although costing a little more in dollars per square foot, also result in a saving because of the reduction in the total form area. These variable skew structures develop some peculiar stresses, especially in the foundations. The theory has been extended to evaluate all the effects of variable skew, and a few tests indicated that the theoretical extension is satisfactory. It would be a great advantage if additional model tests were made using curved-in-plan shapes to supplement the extension of the theory in the paper to these curved-in-plan structures.

Mr. Schwartz's discussion is short but to the point. The solution of simultaneous equations is tedious work, and the results are usually unsatisfactory. There are numerous examples to substantiate this statement, including the discussion by Mr. Sos, who has taken a design[3] that has been checked and rechecked numerous times and has been the subject of much discussion. The inconsistencies mentioned in Mr. Sos' discussion are not unusual. An examination of the equations and the step-by-step solution shows how these errors enter the analysis in such an unobtrusive manner that usually they cannot be detected.

Mr. Arup's discussion is gratifying to the writer in many ways, although it questions the validity of the basic assumptions. It is true that the paper is based on the elastic theory starting with the same assumptions as Mr. Rathbun's original theory. The questions raised by Mr. Arup as to the validity of these assumptions have been raised before[30,31] and the writer can present no better arguments than those written by Messrs. Hodges[46] and Rathbun.[47]

Concerning Fig. 19, it should be observed that for large skews (say 60°) the span at right angles is very short, the area in contact with the support is very wide, and the ratio of width to span is not comparable for beams and bridges. For these conditions it is questionable whether Eq. 52 applies.

To the writer's knowledge, the theory of shells and plates has not been sufficiently developed to apply to skewed rigid frames and arch bridges. Several solutions to the skew plate problem have been published[11,49,33] and suggestions[50,51] have been made to extend the methods to skewed rigid frames and to skewed arch bridges. The solution by finite differences[51] involves the numerical solution of several simultaneous equations. As so ably demonstrated by Mr. Sos, the numerical solution of simultaneous equations for skewed rigid frames often gives erroneous answers. The boundary conditions in skewed rigid frames and arch bridges are much more complicated than in the idealized structures to which these other solutions have been successfully applied. Until these other theories have been perfected and substantiated by test data, the application of the proposed method, which is based on the elastic theory and

[47] "Simplified Analysis of Skewed Reinforced Concrete Frames and Arches," by Richard M. Hodges, *Transactions*, ASCE, Vol. 109, 1944, p. 969.

[48] *Ibid.*, p. 960.

[49] "Structural Skew Plates," by F. L. Ehasz, *Transactions*, ASCE, Vol. 111, 1946, p. 11.

[50] Discussion by Julius Sigman of "Structural Skew Plates," by F. L. Ehasz, *Transactions*, ASCE, Vol. 111, 1946, p. 1042.

[51] "Structural Skew Plates," by F. L. Ehasz, *Ibid.*, p. 1042.

which has been substantiated by comparisons with numerous test data, is recommended. Several engineers have suggested[4,14,49,53,46] that an adequate design procedure might be developed by various lines of attack. All these were explored at great length and all were found inadequate; none yielded a practical design and yet permitted the evaluation of dropped terms; none showed how the limits of approximations could be determined; and none placed a limit on the combined unit shearing stress in the concrete. Exception must be taken to the part of Mr. Arup's concluding statement which maintains that:

> "Since the introduction of the theory of plates and shells,[33,34,36] however, Mr. Rathbun's theory has become obsolete."

A thorough canvass has not yielded a single skew rigid-frame or arch bridge designed by the theory used for plates and shells.

The successful application of this theory to hundreds of actual structures is final proof that the elastic theory is valid and may be applied as a practical office procedure for the design of reinforced concrete skewed rigid-frame and arch bridges.

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[52] "Design of a Reinforced Concrete Skew Arch," by Bernard L. Weiner, *Transactions*, ASCE, Vol. 96, 1942, p. 1286.

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